Mathematical Formulae

A Handbook
with a table of Physical Constants

version 2.6

Compiled for the Physics Department

Contents

Introduction

This Mathematical Formulae handbook has been prepared in response to a request from the Physics Consultative Committee, with the hope that it will be useful to those studying physics. It is to some extent modelled on a similar document issued by the Department of Engineering, but obviously reflects the particular interests of physicists. There was discussion as to whether it should also include physical formulae such as Maxwell's equations, etc., but a decision was taken against this, partly on the grounds that the book would become unduly bulky, but mainly because, in its present form, clean copies can be made available to candidates in exams.

There has been wide consultation among the staff about the contents of this document, but inevitably some users will seek in vain for a formula they feel strongly should be included. Please send suggestions for amendments to the Teaching Committee, and they will be considered for incorporation in the next edition. The Teaching Committee will also be grateful to be informed of any (equally inevitable) errors which are found.

This handbook was compiled by Dr John Shakeshaft and typeset originally by Fergus Gallagher, and currently by Dr Dave Green, using the TEX typesetting package.

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Physical Constants

Based on the 2018 CODATA recommended values. The figures in parentheses give the 1-standard-deviation uncertainties in the last digits. See Newell, D. B. et al. 2018, Metrologia, 55, L13–L16 (doi:10.1088/1681-7575/aa950a).

speed of light in a vacuum	c	$2.997\,924\,58 \times 10^{8}$ m s^{-1}	(by definition)
permeability of a vacuum	μ_0	$1.256\,637\,062\,12(19) \times 10^{-6}$ N A^{-2}	
		$\approx 4\pi \times 10^{-7}$ N A^{-2}	
permittivity of a vacuum	ϵ_0	$8.854\,187\,8128(13) \times 10^{-12}$ F m^{-1}	
elementary charge	e	$1.602\,176\,634 \times 10^{-19}$ C	(by definition)
Planck constant	h	$6.626\,070\,15 \times 10^{-34}$ J s	(by definition)
$h/2\pi$	$\hbar$	$1.054\,571\,817\ldots \times 10^{-34}$ J s	
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$ mol^{-1}	(by definition)
unified atomic mass constant	m_u	$1.660\,539\,066\,60(50) \times 10^{-27}$ kg	
mass of electron	m_e	$9.109\,383\,7015(28) \times 10^{-31}$ kg	
mass of proton	m_p	$1.672\,621\,923\,69(51) \times 10^{-27}$ kg	
Bohr magneton $eh/4\pi m_e$	μ_B	$9.274\,010\,0783(28) \times 10^{-24}$ J T^{-1}	
molar gas constant	R	$8.314\,462\,618\ldots$ J K^{-1} mol^{-1}	
Boltzmann constant	k_B	$1.380\,649 \times 10^{-23}$ J K^{-1}	(by definition)
Stefan–Boltzmann constant	σ	$5.670\,374\,419\ldots \times 10^{-8}$ W m^{-2} K^{-4}	
gravitational constant	G	$6.674\,30(15) \times 10^{-11}$ N m^{2} kg^{-2}	
Other data			
acceleration due to gravity	g	$9.806\,65$ m s^{-2}	(standard value)

1. Series

Arithmetic and Geometric progressions

A.P. $S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-1)d] = \frac{n}{2}[2a+(n-1)d]$

G.P. $S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1-r^n}{1-r}$, $\left(S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1\right)$

(These results also hold for complex series.)

Convergence of series: the ratio test

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \quad \text{converges as} \quad n \to \infty \quad \text{if} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$$

Convergence of series: the comparison test

If each term in a series of positive terms is less than the corresponding term in a series known to be convergent, then the given series is also convergent.

Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

If n is a positive integer the series terminates and is valid for all x: the term in x^r is ${}^nC_r x^r$ or $\binom{n}{r}$ where ${}^nC_r \equiv \frac{n!}{r!(n-r)!}$ is the number of different ways in which an unordered sample of r objects can be selected from a set of n objects without replacement. When n is not a positive integer, the series does not terminate: the infinite series is convergent for $|x| < 1$.

Taylor and Maclaurin Series

If $y(x)$ is well-behaved in the vicinity of $x = a$ then it has a Taylor series,

$$y(x) = y(a+u) = y(a) + u\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{u^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{u^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

where $u = x - a$ and the differential coefficients are evaluated at $x = a$. A Maclaurin series is a Taylor series with $a = 0$,

$$y(x) = y(0) + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{x^3}{3!}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \cdots$$

Power series with real variables

e^x	$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$	valid for all x
$\ln(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$	valid for $-1 < x \le 1$
$\cos x$	$= \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	valid for all values of x
$\sin x$	$= \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$	valid for all values of x
$\tan x$	$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$	valid for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\tan^{-1} x$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$	valid for $-1 \le x \le 1$
$\sin^{-1} x$	$= x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \cdots$	valid for $-1 < x < 1$

Integer series

$$\sum_1^N n = 1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$\sum_1^N n^2 = 1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_1^N n^3 = 1^3 + 2^3 + 3^3 + \cdots + N^3 = [1 + 2 + 3 + \cdots N]^2 = \frac{N^2(N+1)^2}{4}$$

$$\sum_1^\infty \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$ [see expansion of $\ln(1+x)$]

$$\sum_1^\infty \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$ [see expansion of $\tan^{-1} x$]

$$\sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\sum_1^N n(n+1)(n+2) = 1.2.3 + 2.3.4 + \cdots + N(N+1)(N+2) = \frac{N(N+1)(N+2)(N+3)}{4}$$

This last result is a special case of the more general formula,

$$\sum_1^N n(n+1)(n+2)\ldots(n+r) = \frac{N(N+1)(N+2)\ldots(N+r)(N+r+1)}{r+2}.$$

Plane wave expansion

$$\exp(ikz) = \exp(ikr\cos\theta) = \sum_{l=0}^\infty (2l+1) i^l j_l(kr) P_l(\cos\theta),$$

where $P_l(\cos\theta)$ are Legendre polynomials (see section 11) and $j_l(kr)$ are spherical Bessel functions, defined by

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}} J_{l+1/2}(\rho),$$ with $J_l(x)$ the Bessel function of order l (see section 11).

2. Vector Algebra

If $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are orthonormal vectors and $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ then $|\boldsymbol{A}|^2 = A_x^2 + A_y^2 + A_z^2$. [Orthonormal vectors ≡ orthogonal unit vectors.]

Scalar product

$$\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}|\,|\boldsymbol{B}|\cos\theta$$ where θ is the angle between the vectors

$$= A_xB_x + A_yB_y + A_zB_z = [\,A_xA_yA_z\,]\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Scalar multiplication is commutative: $\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{A}$.

Equation of a line

A point $\boldsymbol{r} \equiv (x, y, z)$ lies on a line passing through a point $\boldsymbol{a}$ and parallel to vector $\boldsymbol{b}$ if

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$$

with λ a real number.

Equation of a plane

A point $\boldsymbol{r} \equiv (x, y, z)$ is on a plane if either

(a) $\boldsymbol{r} \cdot \widehat{\boldsymbol{d}} = |\boldsymbol{d}|$, where $\boldsymbol{d}$ is the normal from the origin to the plane, or

(b) $\dfrac{x}{X} + \dfrac{y}{Y} + \dfrac{z}{Z} = 1$ where X, Y, Z are the intercepts on the axes.

Vector product

$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{n}\,|\boldsymbol{A}|\,|\boldsymbol{B}|\sin\theta$, where θ is the angle between the vectors and $\boldsymbol{n}$ is a unit vector normal to the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$ in the direction for which $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{n}$ form a right-handed set of axes.

$\boldsymbol{A} \times \boldsymbol{B}$ in determinant form

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\boldsymbol{A} \times \boldsymbol{B}$ in matrix form

$$\begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

Vector multiplication is not commutative: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$.

Scalar triple product

$$\boldsymbol{A} \times \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -\boldsymbol{A} \times \boldsymbol{C} \cdot \boldsymbol{B}, \quad \text{etc.}$$

Vector triple product

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}, \qquad (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A}$$

Non-orthogonal basis

$$\begin{aligned} \boldsymbol{A} &= A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3 \\ A_1 &= \boldsymbol{\epsilon}' \cdot \boldsymbol{A} \quad \text{where} \quad \boldsymbol{\epsilon}' = \frac{\boldsymbol{e}_2 \times \boldsymbol{e}_3}{\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)} \end{aligned}$$

Similarly for A_2 and A_3.

Summation convention

$$\boldsymbol{a} = a_i\boldsymbol{e}_i \qquad \text{implies summation over } i = 1\ldots 3$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$$

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \varepsilon_{ijk} a_j b_k \qquad \text{where } \varepsilon_{123} = 1; \quad \varepsilon_{ijk} = -\varepsilon_{ikj}$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

3. Matrix Algebra

Unit matrices

The unit matrix I of order n is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If A is a square matrix of order n, then $AI = IA = A$. Also $I = I^{-1}$.
I is sometimes written as I_n if the order needs to be stated explicitly.

Products

If A is a $(n \times l)$ matrix and B is a $(l \times m)$ then the product AB is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik} B_{kj}$$

In general $AB \neq BA$.

Transpose matrices

If A is a matrix, then transpose matrix A^T is such that $(A^T)_{ij} = (A)_{ji}$.

Inverse matrices

If A is a square matrix with non-zero determinant, then its inverse A^{-1} is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of A_{ij} is $(-1)^{i+j}$ times the determinant of the matrix A with the j-th row and i-th column deleted.

Determinants

If A is a square matrix then the determinant of A, $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\ldots} \epsilon_{ijk\ldots} A_{1i} A_{2j} A_{3k} \ldots$$

where the number of the suffixes is equal to the order of the matrix.

2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Product rules

$$(AB \ldots N)^T = N^T \ldots B^T A^T$$

$$(AB \ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1} \qquad \text{(if individual inverses exist)}$$

$$|AB \ldots N| = |A|\,|B| \ldots |N| \qquad \text{(if individual matrices are square)}$$

Orthogonal matrices

An orthogonal matrix Q is a square matrix whose columns q_i form a set of orthonormal vectors. For any orthogonal matrix Q,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

Solving sets of linear simultaneous equations

If A is square then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ if A^{-1} exists, i.e., if $|A| \neq 0$.

If A is square then $A\boldsymbol{x} = 0$ has a non-trivial solution if and only if $|A| = 0$.

An over-constrained set of equations $A\boldsymbol{x} = \boldsymbol{b}$ is one in which A has m rows and n columns, where m (the number of equations) is greater than n (the number of variables). The best solution x (in the sense that it minimizes the error $|A\boldsymbol{x} - \boldsymbol{b}|$) is the solution of the n equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$. If the columns of A are orthonormal vectors then $\boldsymbol{x} = A^T\boldsymbol{b}$.

Hermitian matrices

The Hermitian conjugate of A is $A^\dagger = (A^*)^T$, where A^* is a matrix each of whose components is the complex conjugate of the corresponding components of A. If $A = A^\dagger$ then A is called a Hermitian matrix.

Eigenvalues and eigenvectors

The n eigenvalues λ_i and eigenvectors $\boldsymbol{u}_i$ of an $n \times n$ matrix A are the solutions of the equation $A\boldsymbol{u} = \lambda\boldsymbol{u}$. The eigenvalues are the zeros of the polynomial of degree n, $P_n(\lambda) = |A - \lambda I|$. If A is Hermitian then the eigenvalues λ_i are real and the eigenvectors u_i are mutually orthogonal. $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

$$\operatorname{Tr} A = \sum_i \lambda_i, \quad \text{also } |A| = \prod_i \lambda_i.$$

If S is a symmetric matrix, Λ is the diagonal matrix whose diagonal elements are the eigenvalues of S, and U is the matrix whose columns are the normalized eigenvectors of A, then

$$U^T S U = \Lambda \qquad \text{and} \qquad S = U \Lambda U^T.$$

If x is an approximation to an eigenvector of A then $\boldsymbol{x}^T A\boldsymbol{x}/(\boldsymbol{x}^T\boldsymbol{x})$ (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

Commutators

$$[A, B] \equiv AB - BA$$

$$[A, B] = -[B, A]$$

$$[A, B]^\dagger = [B^\dagger, A^\dagger]$$

$$[A + B, C] = [A, C] + [B, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Hermitian algebra

$$\boldsymbol{b}^\dagger = (b_1^*, b_2^*, \ldots)$$

	Matrix form	Operator form	Bra-ket form
Hermiticity	$\boldsymbol{b}^* \cdot A \cdot \boldsymbol{c} = (A \cdot \boldsymbol{b})^* \cdot \boldsymbol{c}$	$\int \psi^* O \phi = \int (O\psi)^* \phi$	$\langle\psi\|O\|\phi\rangle = \langle\phi\|O^\dagger\|\psi\rangle^*$
Eigenvalues, λ real	$A\boldsymbol{u}_i = \lambda_{(i)}\boldsymbol{u}_i$	$O\psi_i = \lambda_{(i)}\psi_i$	$O\,\|i\rangle = \lambda_i\,\|i\rangle$
Orthogonality	$\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$	$\int \psi_i^* \psi_j = 0$	$\langle i\|j\rangle = 0 \qquad (i \neq j)$
Completeness	$\boldsymbol{b} = \sum_i \boldsymbol{u}_i(\boldsymbol{u}_i \cdot \boldsymbol{b})$	$\phi = \sum_i \psi_i \left(\int \psi_i^* \phi\right)$	$\phi = \sum_i \|i\rangle\,\langle i\|\phi\rangle$
Rayleigh–Ritz			
Lowest eigenvalue	$\lambda_0 \leq \dfrac{\boldsymbol{b}^* \cdot A \cdot \boldsymbol{b}}{\boldsymbol{b}^* \cdot \boldsymbol{b}}$	$\lambda_0 \leq \dfrac{\int \psi^* O \psi}{\int \psi^* \psi}$	$\lambda_0 \leq \dfrac{\langle\psi\|O\|\psi\rangle}{\langle\psi\|\psi\rangle}$

Pauli spin matrices

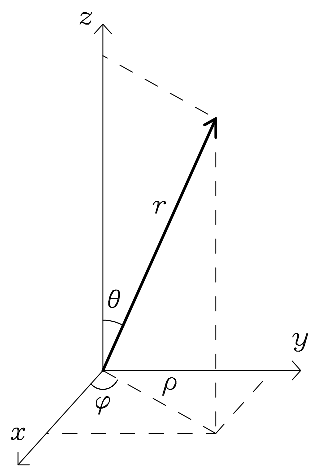
$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_x\sigma_y = \mathrm{i}\sigma_z, \qquad \sigma_y\sigma_z = \mathrm{i}\sigma_x, \qquad \sigma_z\sigma_x = \mathrm{i}\sigma_y, \qquad \sigma_x\sigma_x = \sigma_y\sigma_y = \sigma_z\sigma_z = I$$

4. Vector Calculus

Notation

ϕ is a scalar function of a set of position coordinates. In Cartesian coordinates $\phi = \phi(x, y, z)$; in cylindrical polar coordinates $\phi = \phi(\rho, \varphi, z)$; in spherical polar coordinates $\phi = \phi(r, \theta, \varphi)$; in cases with radial symmetry $\phi = \phi(r)$. $\boldsymbol{A}$ is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates $\boldsymbol{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z$, where A_x, A_y, A_z are independent functions of x, y, z.



In Cartesian coordinates ∇ ('del') $\equiv \boldsymbol{i}\dfrac{\partial}{\partial x} + \boldsymbol{j}\dfrac{\partial}{\partial y} + \boldsymbol{k}\dfrac{\partial}{\partial z} \equiv \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$

$$\operatorname{grad}\phi = \nabla\phi, \qquad \operatorname{div}\boldsymbol{A} = \nabla\cdot\boldsymbol{A}, \qquad \operatorname{curl}A = \nabla\times\boldsymbol{A}$$

Identities

$$\operatorname{grad}(\phi_1 + \phi_2) \equiv \operatorname{grad}\phi_1 + \operatorname{grad}\phi_2 \qquad \operatorname{div}(\boldsymbol{A}_1 + \boldsymbol{A}_2) \equiv \operatorname{div}\boldsymbol{A}_1 + \operatorname{div}\boldsymbol{A}_2$$

$$\operatorname{grad}(\phi_1\phi_2) \equiv \phi_1\operatorname{grad}\phi_2 + \phi_2\operatorname{grad}\phi_1$$

$$\operatorname{curl}(\boldsymbol{A}_1 + \boldsymbol{A}_2) \equiv \operatorname{curl}\boldsymbol{A}_1 + \operatorname{curl}\boldsymbol{A}_2$$

$$\operatorname{div}(\phi\boldsymbol{A}) \equiv \phi\operatorname{div}\boldsymbol{A} + (\operatorname{grad}\phi)\cdot\boldsymbol{A}, \qquad \operatorname{curl}(\phi\boldsymbol{A}) \equiv \phi\operatorname{curl}\boldsymbol{A} + (\operatorname{grad}\phi)\times\boldsymbol{A}$$

$$\operatorname{div}(\boldsymbol{A}_1\times\boldsymbol{A}_2) \equiv \boldsymbol{A}_2\cdot\operatorname{curl}\boldsymbol{A}_1 - \boldsymbol{A}_1\cdot\operatorname{curl}\boldsymbol{A}_2$$

$$\operatorname{curl}(\boldsymbol{A}_1\times\boldsymbol{A}_2) \equiv \boldsymbol{A}_1\operatorname{div}\boldsymbol{A}_2 - \boldsymbol{A}_2\operatorname{div}\boldsymbol{A}_1 + (\boldsymbol{A}_2\cdot\operatorname{grad})\boldsymbol{A}_1 - (\boldsymbol{A}_1\cdot\operatorname{grad})\boldsymbol{A}_2$$

$$\operatorname{div}(\operatorname{curl}\boldsymbol{A}) \equiv 0, \qquad \operatorname{curl}(\operatorname{grad}\phi) \equiv 0$$

$$\operatorname{curl}(\operatorname{curl}\boldsymbol{A}) \equiv \operatorname{grad}(\operatorname{div}\boldsymbol{A}) - \operatorname{div}(\operatorname{grad}\boldsymbol{A}) \equiv \operatorname{grad}(\operatorname{div}\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

$$\operatorname{grad}(\boldsymbol{A}_1\cdot\boldsymbol{A}_2) \equiv \boldsymbol{A}_1\times(\operatorname{curl}\boldsymbol{A}_2) + (\boldsymbol{A}_1\cdot\operatorname{grad})\boldsymbol{A}_2 + \boldsymbol{A}_2\times(\operatorname{curl}\boldsymbol{A}_1) + (\boldsymbol{A}_2\cdot\operatorname{grad})\boldsymbol{A}_1$$

Grad, Div, Curl and the Laplacian

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Cartesian Coordinates		$x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = z$	$x = r\cos\varphi\sin\theta \quad y = r\sin\varphi\sin\theta$ $z = r\cos\theta$
Vector $\boldsymbol{A}$	$A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$	$A_\rho\widehat{\boldsymbol{\rho}} + A_\varphi\widehat{\boldsymbol{\varphi}} + A_z\widehat{\boldsymbol{z}}$	$A_r\widehat{\boldsymbol{r}} + A_\theta\widehat{\boldsymbol{\theta}} + A_\varphi\widehat{\boldsymbol{\varphi}}$
Gradient $\nabla\phi$	$\frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$	$\frac{\partial\phi}{\partial\rho}\widehat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}} + \frac{\partial\phi}{\partial z}\widehat{\boldsymbol{z}}$	$\frac{\partial\phi}{\partial r}\widehat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\widehat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\widehat{\boldsymbol{\varphi}}$
Divergence $\nabla\cdot\boldsymbol{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$
Curl $\nabla\times\boldsymbol{A}$	$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{\rho}\widehat{\boldsymbol{\rho}} & \widehat{\boldsymbol{\varphi}} & \frac{1}{\rho}\widehat{\boldsymbol{z}} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$	$\begin{vmatrix} \frac{1}{r^2\sin\theta}\widehat{\boldsymbol{r}} & \frac{1}{r\sin\theta}\widehat{\boldsymbol{\theta}} & \frac{1}{r}\widehat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & rA_\varphi\sin\theta \end{vmatrix}$
Laplacian $\nabla^2\phi$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$

Transformation of integrals

L = the distance along some curve 'C' in space and is measured from some fixed point.

S = a surface area

τ = a volume contained by a specified surface

$\widehat{\boldsymbol{t}}$ = the unit tangent to C at the point P

$\widehat{\boldsymbol{n}}$ = the unit outward pointing normal

$\boldsymbol{A}$ = some vector function

$\mathrm{d}\boldsymbol{L}$ = the vector element of curve ($= \widehat{\boldsymbol{t}}\,\mathrm{d}L$)

$\mathrm{d}\boldsymbol{S}$ = the vector element of surface ($= \widehat{\boldsymbol{n}}\,\mathrm{d}S$)

Then $\int_C \boldsymbol{A}\cdot\widehat{\boldsymbol{t}}\,\mathrm{d}L = \int_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{L}$

and when $\boldsymbol{A} = \nabla\phi$

$$\int_C (\nabla\phi)\cdot\mathrm{d}\boldsymbol{L} = \int_C \mathrm{d}\phi$$

Gauss's Theorem (Divergence Theorem)

When S defines a closed region having a volume τ

$$\int_\tau (\nabla\cdot\boldsymbol{A})\,\mathrm{d}\tau = \int_S (\boldsymbol{A}\cdot\widehat{\boldsymbol{n}})\,\mathrm{d}S = \int_S \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{S}$$

also

$$\int_\tau (\nabla\phi)\,\mathrm{d}\tau = \int_S \phi\,\mathrm{d}\boldsymbol{S} \qquad \int_\tau (\nabla\times\boldsymbol{A})\,\mathrm{d}\tau = \int_S (\widehat{\boldsymbol{n}}\times\boldsymbol{A})\,\mathrm{d}S$$

Stokes's Theorem

When C is closed and bounds the open surface S,

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} = \int_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{L}$$

also

$$\int_S (\widehat{\boldsymbol{n}} \times \nabla\phi)\ \mathrm{d}S = \int_C \phi\ \mathrm{d}\boldsymbol{L}$$

Green's Theorem

$$\begin{aligned}\int_S \psi\nabla\phi \cdot \mathrm{d}\boldsymbol{S} &= \int_\tau \nabla \cdot (\psi\nabla\phi)\ \mathrm{d}\tau \\ &= \int_\tau \left[\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)\right]\ \mathrm{d}\tau\end{aligned}$$

Green's Second Theorem

$$\int_\tau (\psi\nabla^2\phi - \phi\nabla^2\psi)\ \mathrm{d}\tau = \int_S [\psi(\nabla\phi) - \phi(\nabla\psi)] \cdot \mathrm{d}\boldsymbol{S}$$

5. Complex Variables

Complex numbers

The complex number $z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = r\,\mathrm{e}^{\mathrm{i}(\theta+2n\pi)}$, where $\mathrm{i}^2 = -1$ and n is an arbitrary integer. The real quantity r is the modulus of z and the angle θ is the argument of z. The complex conjugate of z is $z^* = x - \mathrm{i}y = r(\cos\theta - \mathrm{i}\sin\theta) = r\,\mathrm{e}^{-\mathrm{i}\theta}$; $zz^* = |z|^2 = x^2 + y^2$

De Moivre's theorem

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \mathrm{e}^{\mathrm{i}n\theta} = \cos n\theta + \mathrm{i}\sin n\theta$$

Power series for complex variables.

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots \qquad \text{convergent for all finite } z$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad \text{convergent for all finite } z$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad \text{convergent for all finite } z$$

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad \text{principal value of } \ln(1+z)$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the points $z = \pm\mathrm{i}$.

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots$$

This last series converges both on and within the circle $|z| = 1$ except at the point $z = -1$.

6. Trigonometric Formulae

$$\cos^2 A + \sin^2 A = 1 \qquad \sec^2 A - \tan^2 A = 1 \qquad \mathrm{cosec}^2 A - \cot^2 A = 1$$

$$\sin 2A = 2 \sin A \cos A \qquad \cos 2A = \cos^2 A - \sin^2 A \qquad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \qquad \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \qquad \cos^3 A = \frac{3 \cos A + \cos 3A}{4}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \qquad \sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

Relations between sides and angles of any plane triangle

In a plane triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a = b \cos C + c \cos B$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\text{area} = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s = \frac{1}{2}(a+b+c)$$

Relations between sides and angles of any spherical triangle

In a spherical triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

7. Hyperbolic Functions

$$\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{valid for all } x$$

$$\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all } x$$

$\cosh \mathrm{i}x = \cos x$

$\cos \mathrm{i}x = \cosh x$

$\sinh \mathrm{i}x = \mathrm{i}\sin x$

$\sin \mathrm{i}x = \mathrm{i}\sinh x$

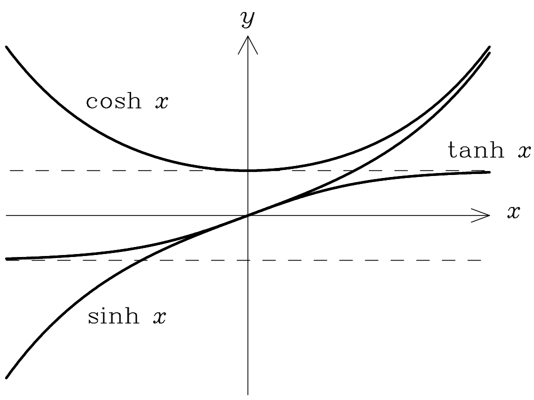
$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$

$\operatorname{cosech} x = \dfrac{1}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$



For large positive x:

$\cosh x \approx \sinh x \to \dfrac{\mathrm{e}^x}{2}$

$\tanh x \to 1$

For large negative x:

$\cosh x \approx -\sinh x \to \dfrac{\mathrm{e}^{-x}}{2}$

$\tanh x \to -1$

Relations of the functions

$\sinh x = -\sinh(-x)$

$\operatorname{sech} x = \operatorname{sech}(-x)$

$\cosh x = \cosh(-x)$

$\operatorname{cosech} x = -\operatorname{cosech}(-x)$

$\tanh x = -\tanh(-x)$

$\coth x = -\coth(-x)$

$$\sinh x = \frac{2\tanh(x/2)}{1-\tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}}$$

$$\cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} = \frac{1}{\sqrt{1-\tanh^2 x}}$$

$\tanh x = \sqrt{1-\operatorname{sech}^2 x}$

$\operatorname{sech} x = \sqrt{1-\tanh^2 x}$

$\coth x = \sqrt{\operatorname{cosech}^2 x + 1}$

$\operatorname{cosech} x = \sqrt{\coth^2 x - 1}$

$\sinh(x/2) = \sqrt{\dfrac{\cosh x - 1}{2}}$

$\cosh(x/2) = \sqrt{\dfrac{\cosh x + 1}{2}}$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$\sinh(2x) = 2\sinh x \cosh x$

$\tanh(2x) = \dfrac{2\tanh x}{1+\tanh^2 x}$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$\sinh(3x) = 3\sinh x + 4\sinh^3 x$

$\cosh 3x = 4\cosh^3 x - 3\cosh x$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

$\sinh x + \sinh y = 2 \sinh\left(\dfrac{x+y}{2}\right) \cosh\left(\dfrac{x-y}{2}\right)$ $\qquad$ $\cosh x + \cosh y = 2 \cosh\left(\dfrac{x+y}{2}\right) \cosh\left(\dfrac{x-y}{2}\right)$

$\sinh x - \sinh y = 2 \cosh\left(\dfrac{x+y}{2}\right) \sinh\left(\dfrac{x-y}{2}\right)$ $\qquad$ $\cosh x - \cosh y = 2 \sinh\left(\dfrac{x+y}{2}\right) \sinh\left(\dfrac{x-y}{2}\right)$

$\sinh x \pm \cosh x = \pm \dfrac{1 \pm \tanh(x/2)}{1 \mp \tanh(x/2)} = \pm\, e^{\pm x}$

$\tanh x \pm \tanh y = \dfrac{\sinh(x \pm y)}{\cosh x \cosh y}$

$\coth x \pm \coth y = \pm \dfrac{\sinh(x \pm y)}{\sinh x \sinh y}$

Inverse functions

$\sinh^{-1} \dfrac{x}{a} = \ln\left(\dfrac{x + \sqrt{x^2 + a^2}}{a}\right)$ $\qquad$ for $-\infty < x < \infty$

$\cosh^{-1} \dfrac{x}{a} = \ln\left(\dfrac{x + \sqrt{x^2 - a^2}}{a}\right)$ $\qquad$ for $x \geq a$

$\tanh^{-1} \dfrac{x}{a} = \dfrac{1}{2} \ln\left(\dfrac{a+x}{a-x}\right)$ $\qquad$ for $x^2 < a^2$

$\coth^{-1} \dfrac{x}{a} = \dfrac{1}{2} \ln\left(\dfrac{x+a}{x-a}\right)$ $\qquad$ for $x^2 > a^2$

$\operatorname{sech}^{-1} \dfrac{x}{a} = \ln\left(\dfrac{a}{x} + \sqrt{\dfrac{a^2}{x^2} - 1}\right)$ $\qquad$ for $0 < x \leq a$

$\operatorname{cosech}^{-1} \dfrac{x}{a} = \ln\left(\dfrac{a}{x} + \sqrt{\dfrac{a^2}{x^2} + 1}\right)$ $\qquad$ for $x \neq 0$

8. Limits

$n^c x^n \to 0$ as $n \to \infty$ if $|x| < 1$ (any fixed c)

$x^n / n! \to 0$ as $n \to \infty$ (any fixed x)

$(1 + x/n)^n \to e^x$ as $n \to \infty$, $x \ln x \to 0$ as $x \to 0$

If $f(a) = g(a) = 0$ then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ (l'Hôpital's rule)

9. Differentiation

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$(uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \cdots + {}^nC_r u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$ Leibniz Theorem

$$\text{where } {}^nC_r \equiv \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \mathrm{sech}^2\, x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, x) = -\,\mathrm{sech}\, x \tanh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\,\mathrm{cosec}^2\, x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\coth x) = -\,\mathrm{cosech}^2\, x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\, x) = -\,\mathrm{cosec}\, x \cot x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosech}\, x) = -\,\mathrm{cosech}\, x \coth x$$

10. Integration

Standard forms

$$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$ for $n \neq -1$

$$\int \frac{1}{x} \,\mathrm{d}x = \ln x + c \qquad \int \ln x \,\mathrm{d}x = x(\ln x - 1) + c$$

$$\int \mathrm{e}^{ax} \,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax} + c \qquad \int x\,\mathrm{e}^{ax} \,\mathrm{d}x = \mathrm{e}^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + c$$

$$\int x \ln x \,\mathrm{d}x = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

$$\int \frac{1}{a^2 + x^2} \,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 - x^2} \,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$$ for $x^2 < a^2$

$$\int \frac{1}{x^2 - a^2} \,\mathrm{d}x = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$$ for $x^2 > a^2$

$$\int \frac{x}{(x^2 \pm a^2)^n} \,\mathrm{d}x = \frac{-1}{2(n-1)}\frac{1}{(x^2 \pm a^2)^{n-1}} + c$$ for $n \neq 1$

$$\int \frac{x}{x^2 \pm a^2} \,\mathrm{d}x = \frac{1}{2}\ln(x^2 \pm a^2) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \,\mathrm{d}x = \ln\left(x + \sqrt{x^2 \pm a^2}\right) + c$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} \,\mathrm{d}x = \sqrt{x^2 \pm a^2} + c$$

$$\int \sqrt{a^2 - x^2} \,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + c$$

$$\int_0^\infty \frac{1}{(1+x)x^p}\,\mathrm{d}x = \pi \operatorname{cosec} p\pi \qquad \text{for } p<1$$

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_{-\infty}^\infty \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \sigma\sqrt{2\pi}$$

$$\int_{-\infty}^\infty x^n \exp(-x^2/2\sigma^2)\,\mathrm{d}x = \begin{cases} 1\times 3\times 5\times\cdots(n-1)\sigma^{n+1}\sqrt{2\pi} & \text{for } n\geq 2 \text{ and even} \\ 0 & \text{for } n\geq 1 \text{ and odd}\end{cases}$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c \qquad \int \sinh x\,\mathrm{d}x = \cosh x + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c \qquad \int \cosh x\,\mathrm{d}x = \sinh x + c$$

$$\int \tan x\,\mathrm{d}x = -\ln(\cos x) + c \qquad \int \tanh x\,\mathrm{d}x = \ln(\cosh x) + c$$

$$\int \operatorname{cosec} x\,\mathrm{d}x = \ln(\operatorname{cosec} x - \cot x) + c \qquad \int \operatorname{cosech} x\,\mathrm{d}x = \ln\left[\tanh(x/2)\right] + c$$

$$\int \sec x\,\mathrm{d}x = \ln(\sec x + \tan x) + c \qquad \int \operatorname{sech} x\,\mathrm{d}x = 2\tan^{-1}(\,\mathrm{e}^x) + c$$

$$\int \cot x\,\mathrm{d}x = \ln(\sin x) + c \qquad \int \coth x\,\mathrm{d}x = \ln(\sinh x) + c$$

$$\int \sin mx \sin nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

$$\int \cos mx \cos nx\,\mathrm{d}x = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + c \qquad \text{if } m^2 \neq n^2$$

Standard substitutions

If the integrand is a function of: substitute:

If the integrand is a function of:	substitute:
(a^2-x^2) or $\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
(x^2+a^2) or $\sqrt{x^2+a^2}$	$x = a\tan\theta$ or $x = a\sinh\theta$
(x^2-a^2) or $\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $x = a\cosh\theta$

If the integrand is a rational function of $\sin x$ or $\cos x$ or both, substitute $t = \tan(x/2)$ and use the results:

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2}.$$

If the integrand is of the form: substitute:

$$\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px+q}} \qquad px+q = u^2$$

$$\int \frac{\mathrm{d}x}{(ax+b)\sqrt{px^2+qx+r}} \qquad ax+b = \frac{1}{u}.$$

Integration by parts

$$\int_a^b u\, \mathrm{d}v = uv\Big|_a^b - \int_a^b v\, \mathrm{d}u$$

Differentiation of an integral

If $f(x, \alpha)$ is a function of x containing a parameter α and the limits of integration a and b are functions of α then

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\, \mathrm{d}x = f(b,\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a,\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)\, \mathrm{d}x.$$

Special case,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(y)\, \mathrm{d}y = f(x).$$

Dirac δ-'function'

$$\delta(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mathrm{i}\omega(t-\tau)]\, \mathrm{d}\omega.$$

If $f(t)$ is an arbitrary function of t then $\int_{-\infty}^{\infty} \delta(t-\tau) f(t)\, \mathrm{d}t = f(\tau)$.

$\delta(t) = 0$ if $t \neq 0$, also $\int_{-\infty}^{\infty} \delta(t)\, \mathrm{d}t = 1$

Reduction formulae

Factorials

$n! = n(n-1)(n-2)\ldots 1, \qquad 0! = 1.$

Stirling's formula for large n: $\quad \ln(n!) \approx n\ln n - n.$

For any $p > -1$, $\int_0^{\infty} x^p\, \mathrm{e}^{-x}\, \mathrm{d}x = p\int_0^{\infty} x^{p-1}\, \mathrm{e}^{-x}\, \mathrm{d}x = p!. \qquad (-1/2)! = \sqrt{\pi}, \qquad (1/2)! = \sqrt{\pi}/2$, etc.

For any $p, q > -1$, $\int_0^1 x^p(1-x)^q\, \mathrm{d}x = \frac{p!q!}{(p+q+1)!}.$

Trigonometrical

If m, n are integers,

$$\int_0^{\pi/2} \sin^m\theta\, \cos^n\theta\, \mathrm{d}\theta = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}\theta\, \cos^n\theta\, \mathrm{d}\theta = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m\theta\, \cos^{n-2}\theta\, \mathrm{d}\theta$$

and can therefore be reduced eventually to one of the following integrals

$$\int_0^{\pi/2} \sin\theta\, \cos\theta\, \mathrm{d}\theta = \frac{1}{2}, \qquad \int_0^{\pi/2} \sin\theta\, \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \cos\theta\, \mathrm{d}\theta = 1, \qquad \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2}.$$

Other

If $I_n = \int_0^{\infty} x^n \exp(-\alpha x^2)\, \mathrm{d}x \quad$ then $\quad I_n = \frac{(n-1)}{2\alpha} I_{n-2}, \quad I_0 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}, \quad I_1 = \frac{1}{2\alpha}.$

11. Differential Equations

Diffusion (conduction) equation

$$\frac{\partial \psi}{\partial t} = \kappa \nabla^2 \psi$$

Wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Legendre's equation

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + l(l+1)y = 0,$$

solutions of which are Legendre polynomials $P_l(x)$, where $P_l(x) = \dfrac{1}{2^l l!}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^l \left(x^2 - 1\right)^l$, *Rodrigues' formula* so $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \dfrac{1}{2}(3x^2 - 1)$ etc.

Recursion relation

$$P_l(x) = \frac{1}{l}\left[(2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)\right]$$

Orthogonality

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\ \mathrm{d}x = \frac{2}{2l+1}\delta_{ll'}$$

Bessel's equation

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2)y = 0,$$

solutions of which are Bessel functions $J_m(x)$ of order m.

Series form of Bessel functions of the first kind

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!(m+k)!} \qquad \text{(integer } m\text{)}.$$

The same general form holds for non-integer $m > 0$.

Laplace's equation

$$\nabla^2 u = 0$$

If expressed in two-dimensional polar coordinates (see section 4), a solution is

$$u(\rho, \varphi) = \left[A\rho^n + B\rho^{-n}\right]\left[C\exp(in\varphi) + D\exp(-in\varphi)\right]$$

where A, B, C, D are constants and n is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

$$u(r, \theta, \varphi) = \left[Ar^l + Br^{-(l+1)}\right]P_l^m\left[C\sin m\varphi + D\cos m\varphi\right]$$

where l and m are integers with $l \geq |m| \geq 0$; A, B, C, D are constants;

$$P_l^m(\cos\theta) = \sin^{|m|}\theta\left[\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}\right]^{|m|} P_l(\cos\theta)$$

is the associated Legendre polynomial.

$$P_l^0(1) = 1.$$

If expressed in cylindrical polar coordinates (see section 4), a solution is

$$u(\rho, \varphi, z) = J_m(n\rho)\left[A\cos m\varphi + B\sin m\varphi\right]\left[C\exp(nz) + D\exp(-nz)\right]$$

where m and n are integers; A, B, C, D are constants.

Spherical harmonics

The normalized solutions $Y_l^m(\theta, \varphi)$ of the equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_l^m + l(l+1)Y_l^m = 0$$

are called spherical harmonics, and have values given by

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^m(\cos\theta)\ \mathrm{e}^{\mathrm{i}m\varphi} \times \begin{cases}(-1)^m & \text{for } m \geq 0\\ 1 & \text{for } m < 0\end{cases}$$

i.e., $Y_0^0 = \sqrt{\frac{1}{4\pi}}$, $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\ \mathrm{e}^{\pm\mathrm{i}\varphi}$, etc.

Orthogonality

$$\int_{4\pi} Y_l^{*m} Y_{l'}^{m'}\ \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}$$

12. Calculus of Variations

The condition for $I = \int_a^b F(y, y', x)\ \mathrm{d}x$ to have a stationary value is $\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)$, where $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. This is the Euler–Lagrange equation.

13. Functions of Several Variables

If $\phi = f(x, y, z, \ldots)$ then $\dfrac{\partial \phi}{\partial x}$ implies differentiation with respect to x keeping $y, z, \ldots$ constant.

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{d}z + \cdots \quad \text{and} \quad \delta\phi \approx \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z + \cdots$$

where $x, y, z, \ldots$ are independent variables. $\dfrac{\partial \phi}{\partial x}$ is also written as $\left(\dfrac{\partial \phi}{\partial x}\right)_{y,\ldots}$ or $\left.\dfrac{\partial \phi}{\partial x}\right|_{y,\ldots}$ when the variables kept constant need to be stated explicitly.

If ϕ is a well-behaved function then $\dfrac{\partial^2 \phi}{\partial x\, \partial y} = \dfrac{\partial^2 \phi}{\partial y\, \partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial \phi}{\partial x}\right)_y = \frac{1}{\left(\dfrac{\partial x}{\partial \phi}\right)_y}, \qquad \left(\frac{\partial \phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial \phi}\right)_x = -1.$$

Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + \frac{1}{2!}\left(u^2\frac{\partial^2 \phi}{\partial x^2} + 2uv\frac{\partial^2 \phi}{\partial x\, \partial y} + v^2\frac{\partial^2 \phi}{\partial y^2}\right) + \cdots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, \quad y = b$

Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$. Unless $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\, \partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left.\begin{array}{ll} \text{Minimum:} & \dfrac{\partial^2 \phi}{\partial x^2} > 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} > 0, \\ \text{Maximum:} & \dfrac{\partial^2 \phi}{\partial x^2} < 0, \text{ or } \dfrac{\partial^2 \phi}{\partial y^2} < 0, \end{array}\right\} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} > \left(\frac{\partial^2 \phi}{\partial x\, \partial y}\right)^2$$

$$\text{Saddle point:} \quad \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} < \left(\frac{\partial^2 \phi}{\partial x\, \partial y}\right)^2$$

If $\dfrac{\partial^2 \phi}{\partial x^2} = \dfrac{\partial^2 \phi}{\partial y^2} = \dfrac{\partial^2 \phi}{\partial x\, \partial y} = 0$ the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If $\phi = f(x, y, \ldots)$ and the variables $x, y, \ldots$ are functions of independent variables $u, v, \ldots$ then

$$\frac{\partial \phi}{\partial u} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial u} + \cdots$$

$$\frac{\partial \phi}{\partial v} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial v} + \cdots$$

etc.

Changing variables in surface and volume integrals – Jacobians

If an area A in the x, y plane maps into an area A' in the u, v plane then

$$\int_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{A'} f(u,v) J\,\mathrm{d}u\,\mathrm{d}v \quad \text{where} \quad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian J is also written as $\frac{\partial(x,y)}{\partial(u,v)}$. The corresponding formula for volume integrals is

$$\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{V'} f(u,v,w) J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \qquad \text{where now} \qquad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

14. Fourier Series and Transforms

Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ then

$$y(x) \approx c_0 + \sum_{m=1}^{M} c_m \cos mx + \sum_{m=1}^{M'} s_m \sin mx$$

where the coefficients are

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(x)\,\mathrm{d}x$$

$$c_m = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\cos mx\,\mathrm{d}x \qquad (m = 1, \ldots, M)$$

$$s_m = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\sin mx\,\mathrm{d}x \qquad (m = 1, \ldots, M')$$

with convergence to $y(x)$ as $M, M' \to \infty$ for all points where $y(x)$ is continuous.

Fourier series for other ranges

Variable t, range $0 \le t \le T$, (i.e., a periodic function of time with period T, frequency $\omega = 2\pi/T$).

$$y(t) \approx c_0 + \sum c_m \cos m\omega t + \sum s_m \sin m\omega t$$

where

$$c_0 = \frac{\omega}{2\pi}\int_0^T y(t)\,\mathrm{d}t, \quad c_m = \frac{\omega}{\pi}\int_0^T y(t)\cos m\omega t\,\mathrm{d}t, \quad s_m = \frac{\omega}{\pi}\int_0^T y(t)\sin m\omega t\,\mathrm{d}t.$$

Variable x, range $0 \le x \le L$,

$$y(x) \approx c_0 + \sum c_m \cos\frac{2m\pi x}{L} + \sum s_m \sin\frac{2m\pi x}{L}$$

where

$$c_0 = \frac{1}{L}\int_0^L y(x)\,\mathrm{d}x, \quad c_m = \frac{2}{L}\int_0^L y(x)\cos\frac{2m\pi x}{L}\,\mathrm{d}x, \quad s_m = \frac{2}{L}\int_0^L y(x)\sin\frac{2m\pi x}{L}\,\mathrm{d}x.$$

Fourier series for odd and even functions

If $y(x)$ is an *odd* (anti-symmetric) function [i.e., $y(-x) = -y(x)$] defined in the range $-\pi \le x \le \pi$, then only sines are required in the Fourier series and $s_m = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} y(x)\sin mx\ \mathrm{d}x$. If, in addition, $y(x)$ is symmetric about $x = \pi/2$, then the coefficients s_m are given by $s_m = 0$ (for m even), $s_m = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} y(x)\sin mx\ \mathrm{d}x$ (for m odd). If $y(x)$ is an *even* (symmetric) function [i.e., $y(-x) = y(x)$] defined in the range $-\pi \le x \le \pi$, then only constant and cosine terms are required in the Fourier series and $c_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} y(x)\ \mathrm{d}x,\quad c_m = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} y(x)\cos mx\ \mathrm{d}x$. If, in addition, $y(x)$ is anti-symmetric about $x = \dfrac{\pi}{2}$, then $c_0 = 0$ and the coefficients c_m are given by $c_m = 0$ (for m even), $c_m = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2} y(x)\cos mx\ \mathrm{d}x$ (for m odd).

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

Complex form of Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ then

$$y(x) \approx \sum_{-M}^{M} C_m\,\mathrm{e}^{\mathrm{i}mx}, \quad C_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} y(x)\,\mathrm{e}^{-\mathrm{i}mx}\ \mathrm{d}x$$

with m taking all integer values in the range $\pm M$. This approximation converges to $y(x)$ as $M \to \infty$ under the same conditions as the real form.

For other ranges the formulae are:

Variable t, range $0 \le t \le T$, frequency $\omega = 2\pi/T$,

$$y(t) = \sum_{-\infty}^{\infty} C_m\,\mathrm{e}^{\mathrm{i}m\omega t}, \quad C_m = \frac{\omega}{2\pi}\int_0^{T} y(t)\,\mathrm{e}^{-\mathrm{i}m\omega t}\ \mathrm{d}t.$$

Variable x', range $0 \le x' \le L$,

$$y(x') = \sum_{-\infty}^{\infty} C_m\,\mathrm{e}^{\mathrm{i}2m\pi x'/L}, \quad C_m = \frac{1}{L}\int_0^{L} y(x')\,\mathrm{e}^{-\mathrm{i}2m\pi x'/L}\ \mathrm{d}x'.$$

Discrete Fourier series

If $y(x)$ is a function defined in the range $-\pi \le x \le \pi$ which is sampled in the $2N$ equally spaced points $x_n = nx/N \quad [n = -(N-1)\dots N]$, then

$$\begin{aligned} y(x_n) = {} & c_0 + c_1\cos x_n + c_2\cos 2x_n + \cdots + c_{N-1}\cos(N-1)x_n + c_N\cos Nx_n \\ & + s_1\sin x_n + s_2\sin 2x_n + \cdots + s_{N-1}\sin(N-1)x_n + s_N\sin Nx_n \end{aligned}$$

where the coefficients are

$$\begin{aligned} c_0 &= \frac{1}{2N}\sum y(x_n) \\ c_m &= \frac{1}{N}\sum y(x_n)\cos mx_n && (m = 1,\dots,N-1) \\ c_N &= \frac{1}{2N}\sum y(x_n)\cos Nx_n \\ s_m &= \frac{1}{N}\sum y(x_n)\sin mx_n && (m = 1,\dots,N-1) \\ s_N &= \frac{1}{2N}\sum y(x_n)\sin Nx_n \end{aligned}$$

each summation being over the $2N$ sampling points x_n.

Fourier transforms

If $y(x)$ is a function defined in the range $-\infty \le x \le \infty$ then the Fourier transform $\widehat{y}(\omega)$ is defined by the equations

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(\omega)\,e^{i\omega t}\,d\omega, \qquad \widehat{y}(\omega) = \int_{-\infty}^{\infty} y(t)\,e^{-i\omega t}\,dt.$$

If ω is replaced by $2\pi f$, where f is the frequency, this relationship becomes

$$y(t) = \int_{-\infty}^{\infty} \widehat{y}(f)\,e^{i2\pi f t}\,df, \qquad \widehat{y}(f) = \int_{-\infty}^{\infty} y(t)\,e^{-i2\pi f t}\,dt.$$

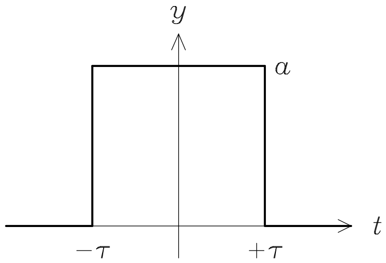
If $y(t)$ is symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega)\cos\omega t\,d\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t)\cos\omega t\,dt.$$

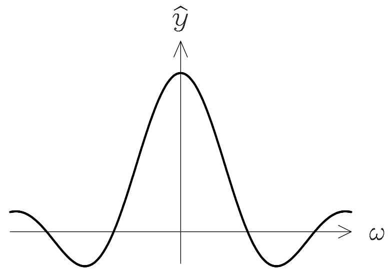
If $y(t)$ is anti-symmetric about $t = 0$ then

$$y(t) = \frac{1}{\pi}\int_{0}^{\infty} \widehat{y}(\omega)\sin\omega t\,d\omega, \qquad \widehat{y}(\omega) = 2\int_{0}^{\infty} y(t)\sin\omega t\,dt.$$

Specific cases

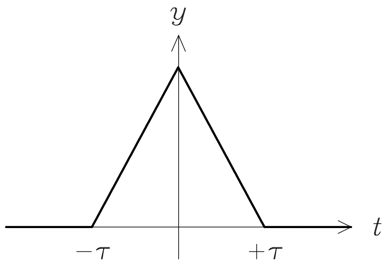


$$\left.\begin{aligned} y(t) &= a, \quad |t| \le \tau \\ &= 0, \quad |t| > \tau \end{aligned}\right\} \quad \textit{('Top Hat')},$$

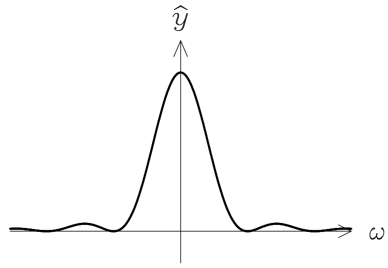


$$\widehat{y}(\omega) = 2a\frac{\sin\omega\tau}{\omega} \equiv 2a\tau\,\mathrm{sinc}(\omega\tau)$$

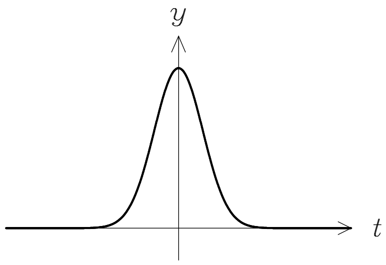
where $\quad \mathrm{sinc}(x) = \dfrac{\sin(x)}{x}$



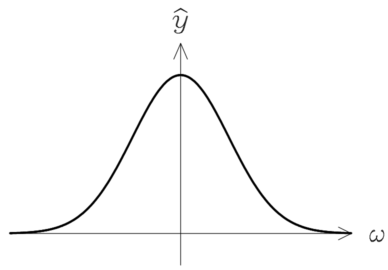
$$\left.\begin{aligned} y(t) &= a(1 - |t|/\tau), \quad &|t| \le \tau \\ &= 0, \quad &|t| > \tau \end{aligned}\right\} \quad \textit{('Saw-tooth')},$$



$$\widehat{y}(\omega) = \frac{2a}{\omega^2\tau}(1 - \cos\omega\tau) = a\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)$$



$$y(t) = \exp(-t^2/t_0^2) \quad \textit{(Gaussian)},$$



$$\widehat{y}(\omega) = t_0\sqrt{\pi}\exp\left(-\omega^2 t_0^2/4\right)$$

$$y(t) = f(t)\,e^{i\omega_0 t} \quad \textit{(modulated function)}, \qquad \widehat{y}(\omega) = \widehat{f}(\omega - \omega_0)$$

$$y(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\tau) \quad \textit{(sampling function)} \qquad \widehat{y}(\omega) = \frac{2\pi}{\tau}\sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/\tau)$$

Convolution theorem

If $z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,\mathrm{d}\tau \quad \equiv x(t) * y(t)$ then $\quad \widehat{z}(\omega) = \widehat{x}(\omega)\,\widehat{y}(\omega)$.

Conversely, $\widehat{xy} = \widehat{x} * \widehat{y}/2\pi$.

Parseval's theorem

$$\int_{-\infty}^{\infty} y^*(t)\,y(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}^*(\omega)\,\widehat{y}(\omega)\,\mathrm{d}\omega$$

(if $\widehat{y}$ is normalised as on page 21)

Fourier transforms in two dimensions

$$\begin{aligned}\widehat{V}(\boldsymbol{k}) &= \int V(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^2\boldsymbol{r} \\ &= \int_0^{\infty} 2\pi r V(r) J_0(kr)\,\mathrm{d}r \qquad \text{if azimuthally symmetric}\end{aligned}$$

Fourier transforms in three dimensions

$$\begin{aligned}\widehat{V}(\boldsymbol{k}) &= \int V(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r} \\ &= \frac{4\pi}{k}\int_0^{\infty} V(r)\,r\sin kr\,\mathrm{d}r \qquad \text{if spherically symmetric} \\ V(\boldsymbol{r}) &= \frac{1}{(2\pi)^3}\int \widehat{V}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{k}\end{aligned}$$

Examples

$V(\boldsymbol{r})$	$\widehat{V}(\boldsymbol{k})$
$\frac{1}{4\pi r}$	$\frac{1}{k^2}$
$\frac{\mathrm{e}^{-\lambda r}}{4\pi r}$	$\frac{1}{k^2+\lambda^2}$
$\nabla V(\boldsymbol{r})$	$\mathrm{i}\boldsymbol{k}\widehat{V}(\boldsymbol{k})$
$\nabla^2 V(\boldsymbol{r})$	$-k^2\widehat{V}(\boldsymbol{k})$

15. Laplace Transforms

If $y(t)$ is a function defined for $t \geq 0$, the Laplace transform $\overline{y}(s)$ is defined by the equation

$$\overline{y}(s) = \mathcal{L}\{y(t)\} = \int_0^\infty \mathrm{e}^{-st} y(t)\, \mathrm{d}t$$

Function $y(t)$ $(t > 0)$	Transform $\overline{y}(s)$	
$\delta(t)$	1	Delta function
$\theta(t)$	$\dfrac{1}{s}$	Unit step function
t^n	$\dfrac{n!}{s^{n+1}}$	
$t^{1/2}$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{s^3}}$	
$t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$	
e^{-at}	$\dfrac{1}{(s+a)}$	
$\sin \omega t$	$\dfrac{\omega}{(s^2+\omega^2}$	
$\cos \omega t$	$\dfrac{s}{(s^2+\omega^2)}$	
$\sinh \omega t$	$\dfrac{\omega}{(s^2-\omega^2)}$	
$\cosh \omega t$	$\dfrac{s}{(s^2-\omega^2)}$	
$\mathrm{e}^{-at} y(t)$	$\overline{y}(s+a)$	
$y(t-\tau)\, \theta(t-\tau)$	$\mathrm{e}^{-s\tau}\, \overline{y}(s)$	
$t y(t)$	$-\dfrac{\mathrm{d}\overline{y}}{\mathrm{d}s}$	
$\dfrac{\mathrm{d}y}{\mathrm{d}t}$	$s\overline{y}(s) - y(0)$	
$\dfrac{\mathrm{d}^n y}{\mathrm{d}t^n}$	$s^n \overline{y}(s) - s^{n-1} y(0) - s^{n-2} \left[\dfrac{\mathrm{d}y}{\mathrm{d}t}\right]_0 \cdots - \left[\dfrac{\mathrm{d}^{n-1}y}{\mathrm{d}t^{n-1}}\right]_0$	
$\int_0^t y(\tau)\, \mathrm{d}\tau$	$\dfrac{\overline{y}(s)}{s}$	
$\left.\begin{array}{l}\int_0^t x(\tau)\, y(t-\tau)\, \mathrm{d}\tau \\ \int_0^t x(t-\tau)\, y(\tau)\, \mathrm{d}\tau\end{array}\right\}$	$\overline{x}(s)\, \overline{y}(s)$	Convolution theorem

[Note that if $y(t) = 0$ for $t < 0$ then the Fourier transform of $y(t)$ is $\widehat{y}(\omega) = \overline{y}(\mathrm{i}\omega)$.]

16. Numerical Analysis

Finding the zeros of equations

If the equation is $y = f(x)$ and x_n is an approximation to the root then either

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{(Newton)}$$

$$\text{or, } x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{(Linear interpolation)}$$

are, in general, better approximations.

Numerical integration of differential equations

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ then

$$y_{n+1} = y_n + hf(x_n, y_n) \quad \text{where } h = x_{n+1} - x_n \qquad \text{(Euler method)}$$

$$\text{Putting} \quad y^*_{n+1} = y_n + hf(x_n, y_n) \qquad \text{(improved Euler method)}$$

$$\text{then} \quad y_{n+1} = y_n + \frac{h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]}{2}$$

Central difference notation

If $y(x)$ is tabulated at equal intervals of x, where h is the interval, then $\delta y_{n+1/2} = y_{n+1} - y_n$ and $\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2}$

Approximating to derivatives

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_n \approx \frac{y_{n+1} - y_n}{h} \approx \frac{y_n - y_{n-1}}{h} \approx \frac{\delta y_{n+\frac{1}{2}} + \delta y_{n-\frac{1}{2}}}{2h} \quad \text{where} \quad h = x_{n+1} - x_n$$

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_n \approx \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} = \frac{\delta^2 y_n}{h^2}$$

Interpolation: Everett's formula

$$y(x) = y(x_0 + \theta h) \approx \overline{\theta} y_0 + \theta y_1 + \frac{1}{3!}\overline{\theta}(\overline{\theta}^2 - 1)\delta^2 y_0 + \frac{1}{3!}\theta(\theta^2 - 1)\delta^2 y_1 + \cdots$$

where θ is the fraction of the interval h $(= x_{n+1} - x_n)$ between the sampling points and $\overline{\theta} = 1 - \theta$. The first two terms represent linear interpolation.

Numerical evaluation of definite integrals

Trapezoidal rule

The interval of integration is divided into n equal sub-intervals, each of width h; then

$$\int_a^b f(x)\,\mathrm{d}x \approx h\left[c\frac{1}{2}f(a) + f(x_1) + \cdots + f(x_j) + \cdots + \frac{1}{2}f(b)\right]$$

where $h = (b-a)/n$ and $x_j = a + jh$.

Simpson's rule

The interval of integration is divided into an even number (say $2n$) of equal sub-intervals, each of width $h = (b-a)/2n$; then

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(b)\right]$$

Gauss's integration formulae

These have the general form $\int_{-1}^{1} y(x)\,\mathrm{d}x \approx \sum_{1}^{n} c_i y(x_i)$

For $n = 2$: $x_i = \pm 0{\cdot}5773$; $c_i = 1, 1$ (exact for any cubic).

For $n = 3$: $x_i = -0{\cdot}7746, 0{\cdot}0, 0{\cdot}7746$; $c_i = 0{\cdot}555, 0{\cdot}888, 0{\cdot}555$ (exact for any quintic).

17. Treatment of Random Errors

Sample mean $\overline{x} = \frac{1}{n}(x_1 + x_2 + \cdots x_n)$

Residual: $d = x - \overline{x}$

Standard deviation of sample: $s = \frac{1}{\sqrt{n}}(d_1^2 + d_2^2 + \cdots d_n^2)^{1/2}$

Standard deviation of distribution: $\sigma \approx \frac{1}{\sqrt{n-1}}(d_1^2 + d_2^2 + \cdots d_n^2)^{1/2}$

Standard deviation of mean:

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{n(n-1)}}(d_1^2 + d_2^2 + \cdots d_n^2)^{1/2}$$
$$= \frac{1}{\sqrt{n(n-1)}}\left[\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2\right]^{1/2}$$

Result of n measurements is quoted as $\overline{x} \pm \sigma_m$.

Range method

A quick but crude method of estimating σ is to find the range r of a set of n readings, i.e., the difference between the largest and smallest values, then

$$\sigma \approx \frac{r}{\sqrt{n}}.$$

This is usually adequate for n less than about 12.

Combination of errors

If $Z = Z(A, B, \ldots)$ (with A, B, etc. independent) then

$$(\sigma_Z)^2 = \left(\frac{\partial Z}{\partial A}\sigma_A\right)^2 + \left(\frac{\partial Z}{\partial B}\sigma_B\right)^2 + \cdots$$

So if

(i) $Z = A \pm B \pm C$, $\quad (\sigma_Z)^2 = (\sigma_A)^2 + (\sigma_B)^2 + (\sigma_C)^2$

(ii) $Z = AB$ or A/B, $\quad \left(\frac{\sigma_Z}{Z}\right)^2 = \left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2$

(iii) $Z = A^m$, $\quad \frac{\sigma_Z}{Z} = m\frac{\sigma_A}{A}$

(iv) $Z = \ln A$, $\quad \sigma_Z = \frac{\sigma_A}{A}$

(v) $Z = \exp A$, $\quad \frac{\sigma_Z}{Z} = \sigma_A$

18. Statistics

Mean and Variance

A random variable X has a distribution over some subset x of the real numbers. When the distribution of X is discrete, the probability that $X = x_i$ is P_i. When the distribution is continuous, the probability that X lies in an interval δx is $f(x)\delta x$, where $f(x)$ is the probability density function.

$$\text{Mean } \mu = E(X) = \sum P_i x_i \text{ or } \int x\, f(x)\, \mathrm{d}x.$$

$$\text{Variance } \sigma^2 = V(X) = E[(X-\mu)^2] = \sum P_i (x_i - \mu)^2 \text{ or } \int (x-\mu)^2 f(x)\, \mathrm{d}x.$$

Probability distributions

Error function: $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \mathrm{e}^{-y^2}\, \mathrm{d}y$

Binomial: $f(x) = \dbinom{n}{x} p^x q^{n-x}$ where $q = (1-p)$, $\quad \mu = np, \sigma^2 = npq, p < 1.$

Poisson: $f(x) = \dfrac{\mu^x}{x!}\mathrm{e}^{-\mu}$, and $\sigma^2 = \mu$

Normal: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2}\right]$

Weighted sums of random variables

If $W = aX + bY$ then $E(W) = aE(X) + bE(Y)$. If X and Y are independent then $V(W) = a^2 V(X) + b^2 V(Y)$.

Statistics of a data sample $x_1, \ldots, x_n$

$$\text{Sample mean } \overline{x} = \frac{1}{n}\sum x_i$$

$$\text{Sample variance } s^2 = \frac{1}{n}\sum (x_i - \overline{x})^2 = \left(\frac{1}{n}\sum x_i^2\right) - \overline{x}^2 = E(x^2) - [E(x)]^2$$

Regression (least squares fitting)

To fit a straight line by least squares to n pairs of points (x_i, y_i), model the observations by $y_i = \alpha + \beta(x_i - \overline{x}) + \epsilon_i$, where the ϵ_i are independent samples of a random variable with zero mean and variance σ^2.

Sample statistics: $s_x^2 = \dfrac{1}{n}\sum (x_i - \overline{x})^2, \quad s_y^2 = \dfrac{1}{n}\sum (y_i - \overline{y})^2, \quad s_{xy}^2 = \dfrac{1}{n}\sum (x_i - \overline{x})(y_i - \overline{y}).$

Estimators: $\widehat{\alpha} = \overline{y}$, $\widehat{\beta} = \dfrac{s_{xy}^2}{s_x^2}$; $E(Y \text{ at } x) = \widehat{\alpha} + \widehat{\beta}(x - \overline{x})$; $\widehat{\sigma}^2 = \dfrac{n}{n-2}$(residual variance),
where residual variance $= \dfrac{1}{n}\sum \{y_i - \widehat{\alpha} - \widehat{\beta}(x_i - \overline{x})\}^2 = s_y^2 - \dfrac{s_{xy}^4}{s_x^2}.$

Estimates for the variances of $\widehat{\alpha}$ and $\widehat{\beta}$ are $\dfrac{\widehat{\sigma}^2}{n}$ and $\dfrac{\widehat{\sigma}^2}{n s_x^2}$.

Correlation coefficient: $\widehat{\rho} = r = \dfrac{s_{xy}^2}{s_x s_y}$.